

Dilaton black holes coupled to nonlinear electrodynamic field

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The theory of nonlinear electrodynamics has got a lot of attentions in recent years. It was shown that Born-Infeld nonlinear electrodynamics is not the only modification of the linear Maxwell's field which keeps the electric field of a charged point particle finite at the origin, and other type of nonlinear Lagrangian such as exponential and logarithmic nonlinear electrodynamics can play the same role. In this paper, we generalize the study on the exponential nonlinear electrodynamics by adding a scalar dilaton field to the action. By suitably choosing the coupling of the matter field to the dilaton field, we vary the action and obtain the corresponding field equations. Then, by making a proper ansatz, we construct a new class of charged dilaton black hole solutions coupled to the exponential nonlinear electrodynamics field in the presence of two Liouville-type potentials for the dilaton field. Due to the presence of the dilaton field, the asymptotic behavior of these solutions are neither flat nor (A)dS. In the limiting case where the nonlinear parameter β^2 goes to infinity, our solution reduces to the Einstein-Maxwell dilaton black holes. We obtain the mass, temperature, entropy and electric potential of these solutions. We also study the behaviour of the electric field as well as the electric potential of these black holes near the origin. We find that the electric field has a finite value *near* the origin, which is the same as the electric field of Born-Infeld nonlinear electrodynamics, but it can diverge exactly at $r = 0$ depending on the model parameters. We also investigate the effects of the dilaton field on the behaviour of the electric field and electric potential. Finally, we check the validity of the first law of black hole thermodynamics on the black hole horizon.

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I. INTRODUCTION

One of the challenges of the classical Maxwell theory is that it leads to an infinite electric field at $r = 0$, and hence an infinite self energy for a charged point particle located at $r = 0$. Although, applying quantum electrodynamics one can remove the divergences in the theory by using the

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renormalization procedure, the problem still remains in the classical electrodynamics. The first attempt to resolve the divergence problem appearing in the Maxwell theory was made by Born and Infeld in 1934 [1]. The Lagrangian of the Born-Infeld (BI) nonlinear gauge field can be written [1]

$$L_{\text{BI}} = 4\beta^2 \left(1 - \sqrt{1 + \frac{F^2}{2\beta^2}} \right), \quad (1)$$

where β is called the nonlinear parameter with dimension of mass, $F^2 = F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ is the electromagnetic field tensor. It has been shown that charged black hole solutions in BI theory are less singular in comparison with the Reissner-Nordström solution. Indeed, in BI theory there is no divergence term q^2/r^2 in the metric near the singularity while the Schwarzschild-type term m/r does exist [2, 3]. In recent years, BI nonlinear electrodynamics has got a new impetus, since it naturally arises in the low energy limit of the open string theory [4, 5]. In addition, most of physical systems in the nature, including the field equations of the gravitational systems, are intrinsically nonlinear and hence including a nonlinear electrodynamics in the action of the gravitational systems is well motivated. Black object solutions coupled to a nonlinear BI gauge field have been studied and their thermodynamics have been investigated in ample details [6–10].

In recent years, other types of nonlinear electrodynamics in the context of gravitational field have been introduced, which can also remove the divergence of the electric field at $r = 0$, similar to BI nonlinear electrodynamics. Two well-known nonlinear Lagrangians for electrodynamics are logarithmic and exponential Lagrangians. Logarithmic nonlinear (LN) $U(1)$ gauge theory was proposed by Soleng [11],

$$L_{\text{LN}} = -4\beta^2 \ln \left(1 + \frac{F^2}{4\beta^2} \right), \quad (2)$$

while the Lagrangian of the exponential nonlinear (EN) electrodynamics was suggested by Hendi as [12],

$$L_{\text{EN}} = 4\beta^2 \left[\exp \left(-\frac{F^2}{4\beta^2} \right) - 1 \right]. \quad (3)$$

It is worth mentioning that logarithmic form of the electrodynamic Lagrangian, like BI electrodynamics, removes divergences in the electric field, while the exponential form of nonlinear electromagnetic field does not cancel the divergency of the electric field at $r = 0$, however, its singularity is much weaker than Einstein-Maxwell theory. Although these two types of nonlinear electrodynamics have no direct relation to superstring theory, they serve as a toy model illustrating that certain nonlinear field theories can produce particle-like solutions which can realize the limiting curvature

hypothesis also for gauge fields [11]. Besides, the expansions of these Lagrangians, for large value of nonlinear parameter β^2 , leads to Maxwell linear Lagrangian, exactly like the BI case,

$$L_{\text{BI}} = L_{\text{LN}} = L_{\text{EN}} = -F^2 + \frac{F^4}{8\beta^2} + O\left(\frac{1}{\beta^4}\right). \quad (4)$$

Clearly, for $\beta^2 \rightarrow \infty$, we arrive at $L_{\text{BI}} = L_{\text{LN}} = L_{\text{EN}} = -F^2$. In addition, from the AdS/CFT correspondence viewpoint in hydrodynamic models, it has been shown that, unlike gravitational correction, higher-derivative terms for Abelian fields in the form of nonlinear electrodynamics do not affect the ratio of shear viscosity over entropy density [14]. Furthermore, in applications of the AdS/CFT correspondence to superconductivity, nonlinear electrodynamic theories make crucial effects on the condensation as well as the critical temperature of the superconductor and its energy gap [15, 16].

For all mentioned above, further studies on EN electrodynamics are well motivated. It is also interesting to extend the study to the dilaton gravity. Of particular interest is to investigate the effects of the dilaton field on the physical properties of the solutions. The appearance of the dilaton field changes the asymptotic behavior of the solutions to be neither asymptotically flat nor (A)dS. There are at least two motivations for investigating non-asymptotically flat, non-asymptotically (A)dS black hole spacetimes. First, these kind of solutions might lead to possible extensions of AdS/CFT correspondence. Indeed, it has been speculated that the linear dilaton spacetimes, which arise as near-horizon limits of dilatonic black holes, might exhibit holography [18]. Second, such kind of solutions may be used to extend the range of validity of methods and tools originally developed for, and tested in the case of, asymptotically flat or asymptotically (A)dS black holes. The BI action including a dilaton and an axion field, appears in the coupling of an open superstring and an Abelian gauge field theory [4]. This action, describing a Born-Infeld-dilaton-axion system coupled to Einstein gravity, can be considered as a nonlinear extension in the Abelian field of Einstein-Maxwell-dilaton-axion gravity. Although one can consistently truncate such models, the presence of the dilaton field cannot be ignored if one considers coupling of the gravity to other gauge fields, and therefore one remains with Einstein-BI gravity in the presence of a dilaton field. Physical properties, thermodynamics and thermal stability of the black object solutions in the context of BI dilaton theory have been investigated [19–28].

In this paper, we turn the investigation on EN electrodynamics by including a dilaton field in the action. When the dilaton field is coupled to the EN gauge field, it has profound consequences for the black hole solutions. Due to the presence of the dilaton field, the asymptotic behaviour of these solutions are neither flat nor (A)dS. We shall investigate the effects of the nonlinearity as well as the

dilaton field on the properties of the solutions. We also compute the conserved and thermodynamic quantities of these solutions and check the validity of the first law of thermodynamics on the black hole horizon.

The organization of this paper is as follows. In Sec. II, we introduce the Lagrangian of EN electrodynamics coupled to the dilaton field in Einstein gravity, and obtain the corresponding field equations by varying the action. Then, we construct a new class of black hole solutions in the presence of two Liouville type potentials for the dilaton field and general dilaton coupling constant. In Sec. III, we investigate the physical properties of the obtained solutions. In Sec. IV, we obtain the conserved and thermodynamic quantities of the spacetime and show that these quantities satisfy the first law of thermodynamics. We finish our paper with some concluding remarks.

II. BASIC EQUATIONS AND SOLUTIONS

Our starting point is the following action in which gravity is coupled to dilaton and nonlinear electrodynamic fields

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} (\mathcal{R} - 2g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - V(\Phi) + L(F, \Phi)), \quad (5)$$

where \mathcal{R} and Φ are, respectively, the Ricci scalar curvature and the dilaton field, and $V(\Phi)$ is the potential for Φ . We choose the Lagrangian of the EN electrodynamics coupled to the dilaton field (ENd), $L(F, \Phi)$, as

$$L(F, \Phi) = 4\beta^2 e^{2\alpha\Phi} \left[\exp \left(-\frac{e^{-4\alpha\Phi} F^2}{4\beta^2} \right) - 1 \right], \quad (6)$$

where α is a constant determining the strength of coupling of the scalar and electromagnetic fields. In order to motivate such a choice for the Lagrangian of ENd field, let us invoke the BI-dilaton (BIId) Lagrangian which is written as [19]

$$L_{\text{BIId}}(F, \Phi) = 4\beta^2 e^{2\alpha\Phi} \left(1 - \sqrt{1 + \frac{e^{-4\alpha\Phi} F^2}{2\beta^2}} \right). \quad (7)$$

Note that Lagrangian (7) originates from the open string version of the BI action coupled to a dilaton field and only valid for the pure electric case [19]. Clearly, this version of the BIId action does not enjoy electric-magnetic duality [23]. This form for the BIId term have been previously investigated by a number of authors [19–28]. Expanding the BIId Lagrangian (7) for large β , leads

to

$$L_{\text{BIId}}(F, \Phi) = -e^{-2\alpha\Phi} F^2 + \frac{e^{-6\alpha\Phi} F^4}{8\beta^2} - \frac{e^{-10\alpha\Phi} F^6}{32\beta^4} + O\left(\frac{1}{\beta^6}\right), \quad (8)$$

while the expansion of (6) for large β , yields

$$L(F, \Phi) = -e^{-2\alpha\Phi} F^2 + \frac{e^{-6\alpha\Phi} F^4}{8\beta^2} - \frac{e^{-10\alpha\Phi} F^6}{96\beta^4} + O\left(\frac{1}{\beta^6}\right). \quad (9)$$

As one can see, the BIId Lagrangian and the ENd Lagrangian given in (6) have similar expansions and so it is interesting to replace (7) with (6) in the action, and investigate the effects of this kind of nonlinear electrodynamics coupled to the dilaton field on the behavior of the solutions. In the absence of the dilaton field ($\alpha = 0$), $L(F, \Phi)$ reduces to (3) as expected. On the other hand, in the limiting case $\beta^2 \rightarrow \infty$, both $L(F, \Phi)$ and $L_{\text{BIId}}(F, \Phi)$ recovers the standard linear Maxwell Lagrangian coupled to the dilaton field [29]

$$L(F, \Phi) = L_{\text{BIId}}(F, \Phi) = -e^{-2\alpha\Phi} F^2. \quad (10)$$

It is convenient to set

$$L(F, \Phi) = 4\beta^2 e^{2\alpha\Phi} \mathcal{L}(Y), \quad (11)$$

where we define

$$\mathcal{L}(Y) = \exp(-Y) - 1, \quad (12)$$

$$Y = \frac{e^{-4\alpha\Phi} F^2}{4\beta^2}. \quad (13)$$

In order to derive the field equations, we vary action (5) with respect to the gravitational field $g_{\mu\nu}$, the dilaton field Φ and the electromagnetic field A_μ . We find

$$\begin{aligned} \mathcal{R}_{\mu\nu} = & 2\partial_\mu\Phi\partial_\nu\Phi + \frac{1}{2}g_{\mu\nu}V(\Phi) - 2e^{-2\alpha\Phi}\partial_Y\mathcal{L}(Y)F_{\mu\eta}F_\nu{}^\eta \\ & + 2\beta^2 e^{2\alpha\Phi} [2Y\partial_Y\mathcal{L}(Y) - \mathcal{L}(Y)] g_{\mu\nu}, \end{aligned} \quad (14)$$

$$\nabla^2\Phi = \frac{1}{4}\frac{\partial V}{\partial\Phi} + 2\alpha\beta^2 e^{2\alpha\Phi} [2Y\partial_Y\mathcal{L}(Y) - \mathcal{L}(Y)], \quad (15)$$

$$\nabla_\mu (e^{-2\alpha\Phi}\partial_Y\mathcal{L}(Y)F^{\mu\nu}) = 0. \quad (16)$$

In case of linear electrodynamics we have $\mathcal{L}(Y) = -Y$, and the system of equations (14)-(16) reduce to the well-known equations of Einstein-Maxwell-dilaton (EMd) gravity [29–32].

In the present work, we search for the static and spherically symmetric black hole solutions of the above field equations. We assume the spacetime metric has the following form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) (d\theta^2 + \sin^2 \theta d\phi^2), \quad (17)$$

where $f(r)$ and $R(r)$ are functions of r which should be determined. The additional function $R(r)$ is introduced in the metric due to the presence of the new degree of freedom associated with the dilation field in the equations of motion. First of all, we integrate the electromagnetic field equation (16). For this purpose we assume all the components of $F_{\mu\nu}$ are zero except F_{tr} :

$$F_{tr} = \frac{qe^{2\alpha\Phi}}{r^2 R^2(r)} \exp \left[-\frac{1}{2} L_W \left(\frac{q^2}{\beta^2 r^4 R^4(r)} \right) \right], \quad (18)$$

where q is an integration constant which is related to the electric charge of the black hole, and $L_W(x) = \text{LambertW}(x)$ is the Lambert function which satisfies [33]

$$L_W(x) e^{L_W(x)} = x, \quad (19)$$

and has the following series expansion

$$L_W(x) = x - x^2 + \frac{3}{2}x^3 - \frac{8}{3}x^4 + \dots \quad (20)$$

Clearly, series (20) converges for $|x| < 1$. Let us note that the electric field in case of BId black holes is given by [27]

$$F_{tr}^{\text{BId}} = \frac{qe^{2\alpha\Phi}}{r^2 R^2(r)} \left[1 + \frac{q^2}{\beta^2 r^4 R^4(r)} \right]^{-1/2}. \quad (21)$$

In the absence of the dilaton field where $\alpha = 0$ and $R(r) = 1$, Eq. (18) reduces to

$$F_{tr} = \frac{q}{r^2} \exp \left[-\frac{1}{2} L_W \left(\frac{q^2}{\beta^2 r^4} \right) \right], \quad (22)$$

while in the limiting case where $\beta \rightarrow \infty$, both F_{tr} and F_{tr}^{BId} reduces to the electric field of EMd black holes [32]

$$F_{tr} = \frac{qe^{2\alpha\Phi}}{r^2 R^2(r)}. \quad (23)$$

The expansion of (18) for large β is given by

$$F_{tr} = \frac{qe^{2\alpha\Phi}}{r^2 R^2(r)} - \frac{1}{2\beta^2} \frac{q^3 e^{2\alpha\Phi}}{r^6 R^6(r)} + O\left(\frac{1}{\beta^4}\right). \quad (24)$$

In order to construct exact analytical solutions of the field equation (14) and (15) for an arbitrary dilaton coupling constant α , we assume the dilaton potential contain two Liouville terms,

$$V(\Phi) = 2\Lambda_0 e^{2\zeta_0\Phi} + 2\Lambda e^{2\zeta\Phi}, \quad (25)$$

where Λ_0 , Λ , ζ_0 and ζ are constants. This kind of potential was previously investigated in the context of EMD gravity [29, 32] as well as BId black holes [26, 27]. The system of equations (14) and (15) contain three unknown functions $f(r)$, $R(r)$ and $\Phi(r)$. In order to solve them, we make the ansatz

$$R(r) = e^{\alpha\Phi}. \quad (26)$$

This ansatz was first introduced in [34] for the purpose of finding black string solutions of EMD gravity, and latter was applied for constructing black hole solutions of nonlinear BId theory [27]. It is important to note that in the absence of the dilaton field ($\alpha = 0$), we have $R(r) = 1$, as one expected (see Eq. 17). Substituting (26), the electromagnetic field (18) and the metric (17) into the field equations (14) and (15), one can obtain the following solutions

$$f(r) = -\frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-\gamma} r^\gamma - \frac{m}{r^{1-\gamma}} + \frac{(\Lambda + 2\beta^2)(\alpha^2 + 1)^2 b^\gamma}{\alpha^2 - 3} r^{2-\gamma} - \frac{2\beta q}{r^{1-\gamma}} (\alpha^2 + 1) b^\gamma \int r^{-\gamma} \left(\sqrt{L_W(\eta)} - \frac{1}{\sqrt{L_W(\eta)}} \right) dr, \quad (27)$$

$$\Phi(r) = \frac{\alpha}{\alpha^2 + 1} \ln \left(\frac{b}{r} \right), \quad (28)$$

where b is an arbitrary constant, $\gamma = 2\alpha^2/(1 + \alpha^2)$, and

$$\eta \equiv \frac{q^2 r^{2\gamma-4}}{\beta^2 b^{2\gamma}}. \quad (29)$$

In the above expression, m appears as an integration constant and is related to the Arnowitt-Deser-Misner (ADM) mass of the black hole. The obtained solutions fully satisfy the system of equations (14) and (15) provided we take

$$\zeta_0 = \frac{1}{\alpha}, \quad \zeta = \alpha, \quad \Lambda_0 = \frac{b^{-2}\alpha^2}{\alpha^2 - 1}. \quad (30)$$

Notice that Λ remains as a free parameter which plays the role of the cosmological constant. One can redefine it as $\Lambda = -3/l^2$, where l is a constant with dimension of length. The integral of Eq. (27) can be performed using the Mathematica software. The resulting solution can be written

$$f(r) = -\frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-\gamma} r^\gamma - \frac{m}{r^{1-\gamma}} + \frac{(\Lambda + 2\beta^2)(\alpha^2 + 1)^2 b^\gamma}{\alpha^2 - 3} r^{2-\gamma} + \frac{\beta q (\alpha^2 + 1)^4}{2(\alpha^2 - 1)^2} \left(\frac{\beta^2 b^{2\gamma}}{q^2} \right)^{\frac{1-\gamma}{2\gamma-4}} r^{\gamma-1} \times \left(\frac{1 - \alpha^2}{4} \right)^{\frac{2\gamma-3}{2\gamma-4}} \left\{ -4(\gamma - 2)^2 \left[\Gamma \left(\frac{\alpha^2 + 5}{4}, \frac{1 - \alpha^2}{4} L_W(\eta) \right) - \Gamma \left(\frac{\alpha^2 + 5}{4} \right) \right] + (\gamma - 1)^2 \left[\Gamma \left(\frac{\alpha^2 - 3}{4}, \frac{1 - \alpha^2}{4} L_W(\eta) \right) - \Gamma \left(\frac{\alpha^2 - 3}{4} \right) \right] \right\}. \quad (31)$$

where $\Gamma(a, z)$ and $\Gamma(a)$ are Gamma functions and they are related to each other as,

$$\Gamma(a, z) = \Gamma(a) - \frac{z^a}{a} F(a, 1+a, -z). \quad (32)$$

where $F(a, b, z)$ is hypergeometric function [33]. Using (32), solution (31) can also be expressed in terms of hypergeometric function,

$$\begin{aligned} f(r) = & -\frac{\alpha^2+1}{\alpha^2-1} b^{-\gamma} r^\gamma - \frac{m}{r^{1-\gamma}} + \frac{(\Lambda+2\beta^2)(\alpha^2+1)^2 b^\gamma}{\alpha^2-3} r^{2-\gamma} + 2\beta q (\alpha^2+1)^2 r^{\gamma-1} \left(\frac{\beta^2 b^{2\gamma}}{q^2}\right)^{\frac{1-\gamma}{2\gamma-4}} L_W^{\frac{3-2\gamma}{2\gamma-4}}(\eta) \\ & \times \left\{ \frac{L_W^2(\eta)}{\alpha^2+5} F\left(\frac{\alpha^2+5}{4}, \frac{\alpha^2+9}{4}, \frac{\alpha^2-1}{4} L_W(\eta)\right) - \frac{1}{\alpha^2-3} F\left(\frac{\alpha^2-3}{4}, \frac{\alpha^2+1}{4}, \frac{\alpha^2-1}{4} L_W(\eta)\right) \right\}. \end{aligned} \quad (33)$$

In the absence of the dilaton field ($\alpha = 0 = \gamma$), our solution reduces to the one obtained in [13] for asymptotic Reissner-Nordström (RN) black hole coupled to EN electrodynamics in AdS spaces. Using the fact that $L_W(x)$ has a convergent series expansion for $|x| < 1$ as given in (20), we can expand (31) for large β . We find

$$\begin{aligned} f(r) = & -\frac{\alpha^2+1}{\alpha^2-1} b^{-\gamma} r^\gamma - \frac{m}{r^{1-\gamma}} + \frac{\Lambda(\alpha^2+1)^2}{\alpha^2-3} b^\gamma r^{2-\gamma} \\ & + \frac{(\alpha^2+1)b^{-\gamma} q^2}{r^{2-\gamma}} - \frac{q^4(\alpha^2+1)^2 b^{-3\gamma}}{4\beta^2(\alpha^2+5)r^{6-3\gamma}} + O\left(\frac{1}{\beta^4}\right). \end{aligned} \quad (34)$$

This is exactly the result obtained in [27] for BId black holes in the limit of large β . This is an expected result, since as we discussed already, in the limit of large β , the Lagrangian of BId and Lagrangian of ENd electrodynamics have the same expansion and thus the resulting solutions have the same behavior too. In the absence of the dilaton field ($\alpha = 0 = \gamma$), solution (34) reduces to

$$f(r) = 1 - \frac{m}{r} - \frac{\Lambda}{3} r^2 + \frac{q^2}{r^2} - \frac{1}{20\beta^2} \frac{q^4}{r^6} + O\left(\frac{1}{\beta^4}\right), \quad (35)$$

which has the form of static spherically symmetric RN black hole in AdS spacetime in the limit $\beta \rightarrow \infty$. The last term in the right hand side of (35) is the leading nonlinear correction term to the RN-AdS black hole in the large β limit.

III. PHYSICAL PROPERTIES OF SOLUTIONS

In this section we would like to investigate the physical properties of the solutions. For this purpose, we first study the behavior of the electric field of the obtained solution. Combining Eqs. (26) and (28) with (18), we find

$$F_{tr} = \frac{q}{r^2} \exp\left[-\frac{1}{2} L_W\left(\frac{q^2 r^{2\gamma-4}}{\beta^2 b^{2\gamma}}\right)\right]. \quad (36)$$

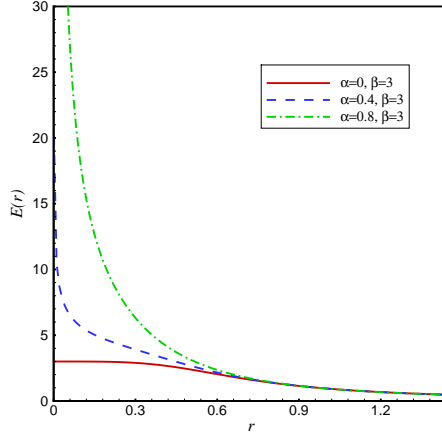


FIG. 1: The behavior of the electric field $E(r)$ of BI-d black holes versus r for $b = 1$ and $q = 1$.

Expanding for large β , we arrive at

$$F_{tr} = \frac{q}{r^2} - \frac{q^3 b^{-2\gamma}}{2\beta^2 r^{6-2\gamma}} + O\left(\frac{1}{\beta^4}\right). \quad (37)$$

We have plotted the behavior of the electric field versus r in Figs. 1-5. From these figures we see that in all cases the electric field goes to zero for large r independent of the value of the other parameters. Figure 1 shows that for BI-d black holes, and in the absence of the dilaton field ($\alpha = 0$), the electric field has a finite value at $r = 0$, while as soon as the dilaton field is taken into account ($\alpha > 0$), the electric field diverges as $r \rightarrow 0$. The behavior of the electric fields for EN-d black holes and different value of α is shown in figure 2. From this figure we see that for $\alpha = 0$, the electric field has a finite value *near* the origin, while it diverges exactly at $r = 0$. This is in contrast to the BI electrodynamics. Again, with increasing α , the divergency of the electric field increases near the origin where $r \rightarrow 0$. In figures 3 and 4 we have compared the behavior of $E(r)$ for BI-d, EN-d and EM-d black holes for both $\alpha = 0$ and $\alpha = 0.4$. Finally, we have plotted in figure 5 the electric field of EN-d black holes for different values of the nonlinear parameter β . From this figure we see that with increasing β , the electric field diverges near the origin. This is an expected result, since for large β our theory reduces to the well-known EM-d gravity [32].

Next, we look for the curvature singularities and horizons. In the presence of the dilaton field, the Kretschmann scalar $R_{\mu\nu\lambda\kappa}R^{\mu\nu\lambda\kappa}$ diverges as $r \rightarrow 0$. Thus, the spacetime has an essential singularity at $r = 0$. However, the spacetime is neither asymptotically flat nor (A)dS. As one can see from Eq. (27), the solution is ill-defined for $\alpha^2 = 3$ and $\alpha^2 = 1$. Now, we search for the horizons of spacetime. The horizons, if any exist can be obtained by solving $f(r_+) = 0$. Clearly, due to the complexity of $f(r)$ given in (31), it is not possible to find the roots of $f(r_+) = 0$, analytically.

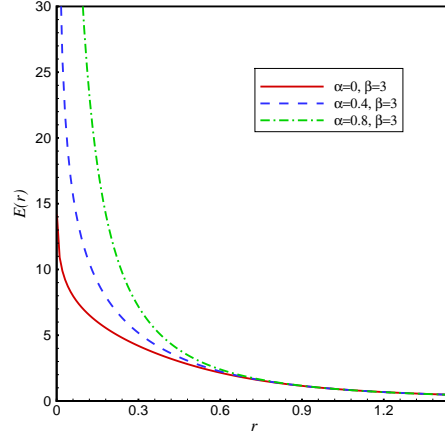


FIG. 2: The behavior of the electric field $E(r)$ of ENd black holes versus r for $b = 1$ and $q = 1$.

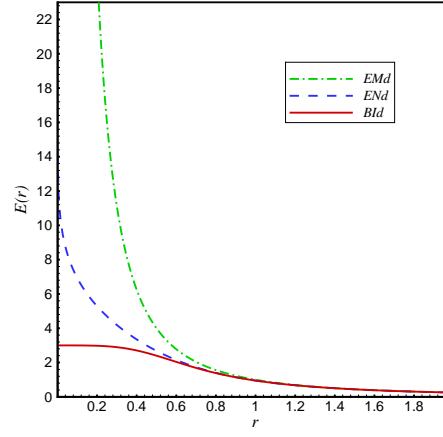


FIG. 3: The behavior of the electric field $E(r)$ versus r for $b = 1$, $\beta = 3$, $\alpha = 0$ and $q = 1$.

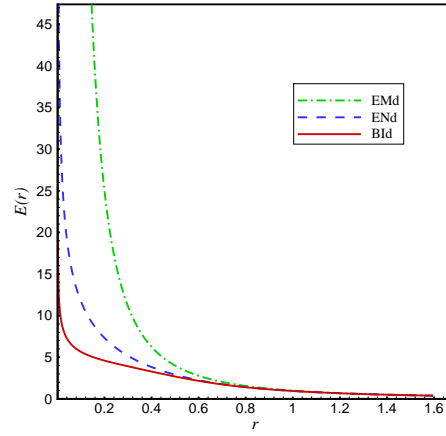


FIG. 4: The behavior of the electric field $E(r)$ versus r for $b = 1$, $\beta = 3$, $\alpha = 0.4$ and $q = 1$.

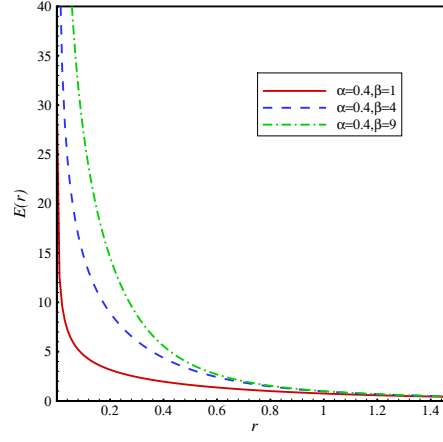


FIG. 5: The behavior of the electric field $E(r)$ of ENd black holes versus r for $b = 1$ and $q = 1$.

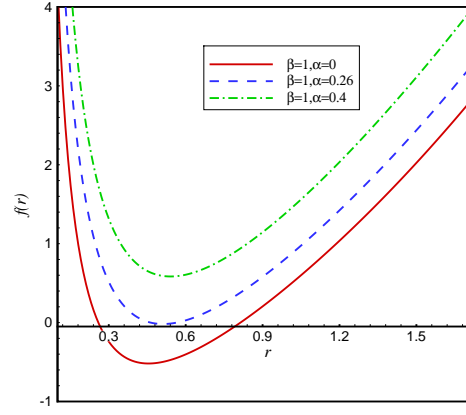


FIG. 6: $f(r)$ versus r for $q = 1$ and $m = 2.5$.

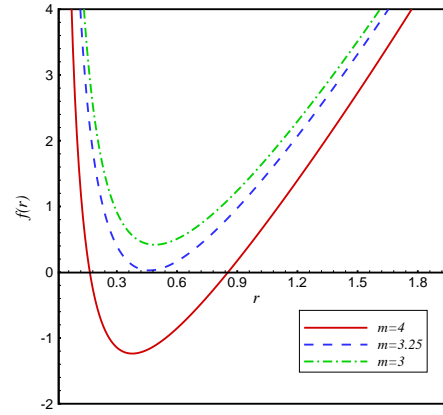


FIG. 7: $f(r)$ versus r for $q = 1$, $\alpha = 0.5$ and $\beta = 1$.

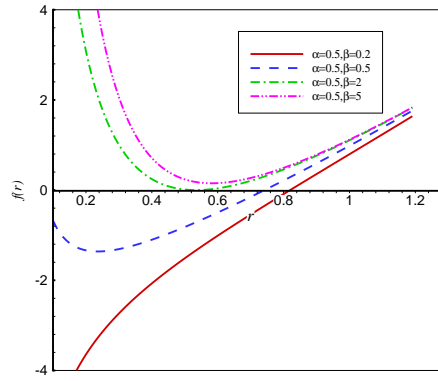


FIG. 8: $f(r)$ versus r for $m = 3.5$ and $\alpha = 0.5$ and $q = 1$.

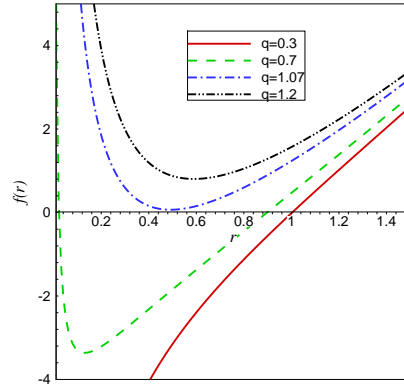


FIG. 9: $f(r)$ versus r for $m = 3.5$ and $\alpha = 0.5$ and $\beta = 1$.

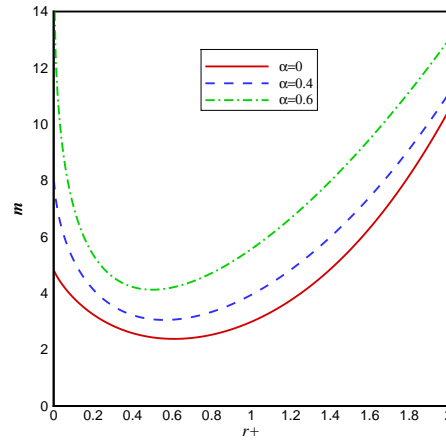


FIG. 10: The mass parameter m versus r_+ for $\beta = 2$ and $q = 1$.

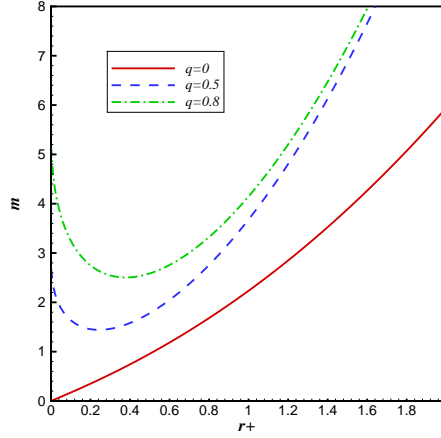


FIG. 11: The mass parameter m versus r_h for $\alpha = 0.5$ and $\beta = 1$.

However, we have plotted the function $f(r)$ versus r for different model parameters in figures 6-9. For simplicity, in these figures, we kept fixed the other parameters $l = b = 1$. These figures show that the obtained solutions may represent a black hole with two horizons, an extreme black hole or a naked singularity depending on the metric parameters. For example, figure 6 shows that for fixed value of the other parameters, the number of horizons decreases with increasing α , while one can see from figure 7 that with increasing m , the number of horizons increases. Figure 8 shows that for fixed value of m , α and q , there is a minimum (extreme) value for the nonlinear parameter β_{\min} (β_{ext}), for which we have black hole with a non-extreme horizon provided $\beta \leq \beta_{\min}$, black hole with two horizons for $\beta_{\min} < \beta < \beta_{\text{ext}}$, black hole with an extreme horizon for $\beta = \beta_{\text{ext}}$ and naked singularity for $\beta > \beta_{\text{ext}}$. Clearly, β_{\min} and β_{ext} depend on the other parameters of the model. Similar, argument can be applied for the charge parameter, q , which is shown in figure 9.

To have further understanding on the nature of the horizons, we plot in figures 10 and 11, the mass parameter m as a function of the horizon radius r_+ for different model parameters. Again, we set $l = b = 1$, for simplicity. These figures show that, for fixed value of other parameters, the value of m determines the number of horizons. We see that, up to a certain value of the mass parameter m , there are two horizons, and as we decrease m further, the two horizons meet. In this case we get extremal black hole (see the next section). Using the fact that $f(r_+) = 0$, we can obtain the mass parameter in terms of the horizon radius,

$$m(r_+) = -\frac{\alpha^2 + 1}{\alpha^2 - 1} b^{-\gamma} r_+ + \frac{(\Lambda + 2\beta^2)(\alpha^2 + 1)^2 b^\gamma}{\alpha^2 - 3} r_+^{3-2\gamma} + 2\beta q (\alpha^2 + 1)^2 \left(\frac{\beta^2 b^{2\gamma}}{q^2} \right)^{\frac{1-\gamma}{2\gamma-4}} L_W^{\frac{3-2\gamma}{2\gamma-4}}(\eta_+) \\ \times \left\{ \frac{L_W^2(\eta_+)}{\alpha^2 + 5} F\left(\frac{\alpha^2 + 5}{4}, \frac{\alpha^2 + 9}{4}, \frac{\alpha^2 - 1}{4} L_W(\eta_+)\right) - \frac{1}{\alpha^2 - 3} F\left(\frac{\alpha^2 - 3}{4}, \frac{\alpha^2 + 1}{4}, \frac{\alpha^2 - 1}{4} L_W(\eta_+)\right) \right\},$$

where $\eta_+ = \eta(r = r_+)$. Figures 11 also shows that in the limit $r_+ \rightarrow 0$ we have a nonzero value for the mass parameter m . This is in contrast to the Schwarzschild black holes in which mass parameter goes to zero as $r_+ \rightarrow 0$. This is due to the effect of the nonlinearity of the electrodynamic field and in case of $q = 0$, the mass parameter m goes to zero as $r_+ \rightarrow 0$.

IV. CONSERVED AND THERMODYNAMICS QUANTITIES

In this section we want to compute the conserved and thermodynamic quantities of the ENd black hole solutions we just found. There are several ways for calculating the mass of the black holes. For example, for asymptotically AdS solution one can use the counterterm method inspired by (A)dS/CFT correspondence [35, 36]. Another way for calculating the mass is through the use of the subtraction method of Brown and York [37]. Such a procedure causes the resulting physical quantities to depend on the choice of reference background. In our case, due to the presence of the non-trivial dilaton field, the asymptotic behaviour of the solutions are neither flat nor (A)dS, therefore we have used the reference background metric and calculate the mass. According to the subtraction method of [37], if we write the metric of static spherically symmetric spacetime in the form [29]

$$ds^2 = -W^2(r)dt^2 + \frac{dr^2}{V^2(r)} + r^2 d\Omega^2, \quad (39)$$

and the matter action contains no derivatives of the metric, then the quasilocal mass is given by [29]

$$\mathcal{M} = rW(r) (V_0(r) - V(r)). \quad (40)$$

Here $V_0(r)$ is an arbitrary function which determines the zero of the energy for a background spacetime and r is the radius of the spacelike hypersurface boundary. It was argued that the ADM mass M is the \mathcal{M} determined in (40) in the limit $r \rightarrow \infty$ [29]. Transforming metric (17) in the form (39), the mass of the ENd black hole is obtained as

$$M = \frac{b^\gamma m \omega}{8\pi(\alpha^2 + 1)}. \quad (41)$$

where ω is the area of an unit 2-sphere. One can obtain the temperature of the horizon by analytic continuation of the metric. The analytical continuation of the Lorentzian metric by $t \rightarrow i\tau$ yields the Euclidean section, whose regularity at $r = r_+$ requires that we should identify $\tau \sim \tau + \beta_+$,

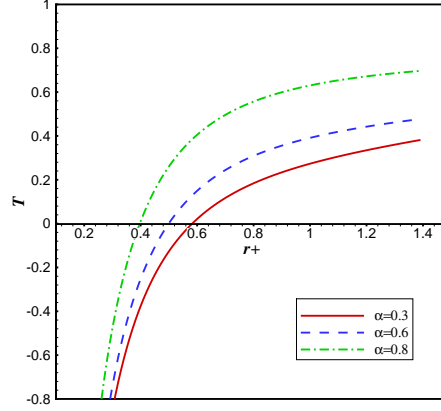


FIG. 12: T versus r_+ for $\beta = 2$, $\Lambda = -3$ and $q = 1$.

where the period $\beta_+ = 1/T$ is the inverse Hawking temperature T of the horizon. It is a matter of calculation to show that

$$T_+ = \frac{1}{4\pi} \left(\frac{df(r)}{dr} \right)_{r=r_+} = -\frac{(\alpha^2 + 1)}{4\pi} r_+^{1-\gamma} \left\{ \frac{b^{-\gamma} r_+^{2\gamma-2}}{\alpha^2 - 1} + (\Lambda + 2\beta^2) b^\gamma - 2\beta q r_+^{\gamma-2} \left(\frac{1}{\sqrt{L_W(\eta_+)}} - \sqrt{L_W(\eta_+)} \right) \right\}, \quad (42)$$

where $\eta_+ = \eta(r = r_+)$ and we have used $f(r_+) = 0$. The behavior of T versus r_+ is shown in figures 12 and 13. From these figures we find out that, for large value of r_+ , the temperature tends to a constant independent of the model parameters. On the other hand, for small values of r_+ , the temperature may be negative ($T < 0$). In this case we encounter a naked singularity. The temperature is zero and the horizon is degenerate for an extremal black hole. In this case r_{ext} is the positive root of the following equation:

$$\frac{b^{-2\gamma} r_{\text{ext}}^{2\gamma-2}}{\alpha^2 - 1} + \Lambda + 2\beta^2 - 2\beta q_{\text{ext}} b^{-\gamma} r_{\text{ext}}^{\gamma-2} \left(\frac{1}{\sqrt{L_W(\eta_{\text{ext}})}} - \sqrt{L_W(\eta_{\text{ext}})} \right) = 0. \quad (43)$$

where

$$\eta_{\text{ext}} \equiv \frac{q_{\text{ext}}^2 r_{\text{ext}}^{2\gamma-4}}{\beta^2 b^{2\gamma}}. \quad (44)$$

From figures 12 and 13, we see that r_{ext} decreases as α increases, while r_{ext} increases with increasing q . Indeed, the metric of Eqs. (17) and (31) can describe a nonlinear dilaton black hole with inner and outer event horizons located at r_- and r_+ , provided $r > r_{\text{ext}}$, an extreme ENd black hole in case of $r = r_{\text{ext}}$, and a naked singularity if $r < r_{\text{ext}}$. Note that in the limiting case where $\beta \rightarrow \infty$, expression (42) reduces to that of black hole in EMD theory [32]

$$T_+ = -\frac{b^{-\gamma}(\alpha^2 + 1)}{8\pi(\alpha^2 - 1)} r_+^{\gamma-1} - \frac{\Lambda(\alpha^2 + 1)b^\gamma}{4\pi} r_+^{1-\gamma} - \frac{q^2 b^{-\gamma}(\alpha^2 + 1)}{4\pi} r_+^{\gamma-3}$$

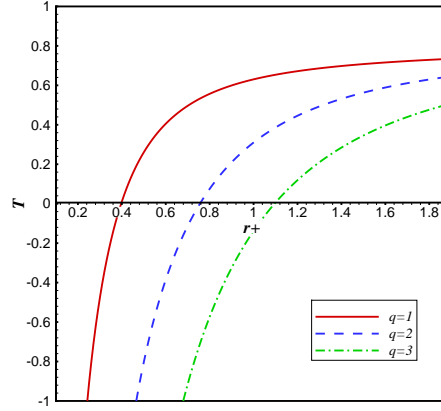


FIG. 13: T versus r_+ for $\alpha = 0.8$, $\Lambda = -3$ and $\beta = 2$.

$$+ \frac{q^4 b^{-3\gamma} (\alpha^2 + 1)}{16\pi\beta^2} r_+^{3\gamma-7} + O\left(\frac{1}{\beta^4}\right). \quad (45)$$

The entropy of the black hole satisfies the so called area law of the entropy which states that the entropy of the black hole is a quarter of the event horizon area [38]. This near universal law applies to almost all kinds of black holes, including dilaton black holes, in Einstein gravity [39]. It is easy to show

$$S = \frac{A}{4} = \frac{b\gamma r_+^{2-\gamma} \omega}{4}. \quad (46)$$

Using the Gauss's law, we can calculate the flux of the electromagnetic field at infinity to obtain the electric charge of black hole as

$$Q = \frac{1}{4\pi} \int e^{-2\alpha\Phi} *F d\Omega = \frac{q\omega}{4\pi}. \quad (47)$$

The electric potential U , measured at infinity with respect to the horizon, is defined by

$$U = A_\mu \chi^\mu|_{r \rightarrow \infty} - A_\mu \chi^\mu|_{r=r_+}, \quad (48)$$

where $\chi = \partial_t$ is the null generator of the horizon. The gauge potential A_t corresponding to the electromagnetic field (37) can be written as

$$A_t = b^\gamma \beta (\alpha^2 + 1) \left(\frac{\beta b^\gamma}{q} \right)^{\frac{1-\gamma}{\gamma-2}} \left(\frac{1-\alpha^2}{4} \right)^{\frac{1}{2\gamma-4}} \\ \times \left\{ -\frac{1}{4} \Gamma\left(\frac{\alpha^2+1}{4}, \frac{1-\alpha^2}{4} L_W(\eta)\right) + \frac{1}{\alpha^2-1} \left[\Gamma\left(\frac{\alpha^2+5}{4}, \frac{1-\alpha^2}{4} L_W(\eta)\right) - \frac{1}{2} \Gamma\left(\frac{\alpha^2+1}{4}\right) \right] \right\}. \quad (49)$$

Therefore, the electric potential may be obtained as

$$U = b^\gamma \beta (\alpha^2 + 1) \left(\frac{\beta b^\gamma}{q} \right)^{\frac{1-\gamma}{\gamma-2}} \left(\frac{1-\alpha^2}{4} \right)^{\frac{1}{2\gamma-4}} \times \left\{ -\frac{1}{4} \Gamma \left(\frac{\alpha^2+1}{4}, \frac{1-\alpha^2}{4} L_W(\eta_+) \right) + \frac{1}{\alpha^2-1} \left[\Gamma \left(\frac{\alpha^2+5}{4}, \frac{1-\alpha^2}{4} L_W(\eta_+) \right) - \frac{1}{2} \Gamma \left(\frac{\alpha^2+1}{4} \right) \right] \right\}. \quad (50)$$

Expanding for large value of β , we get

$$U = \frac{q}{r_+} - \frac{b^{-2\gamma}(\alpha^2+1)}{2\beta^2(\alpha^2+5)} \frac{q^3}{r_+^{5-2\gamma}} + O\left(\frac{1}{\beta^4}\right). \quad (51)$$

We have shown the behavior of the electric potential U as a function of horizon radius r_+ in figures 14-16 for $b = 1$. Due to the nature of the nonlinear electrodynamics, the electric potential can be finite as $r_+ \rightarrow 0$, depending on the model parameters, and goes to zero for large r_+ independent of the model parameters. From these figures we find that for fixed value of other parameter, the divergency of U , for small r_+ , increases with increasing α and β .

Having the conserved and thermodynamic quantities at hand, we are in a position to check the first law of thermodynamics for the obtained solutions. For this purpose, we first obtain the mass M as a function of extensive quantities S and Q . Combining expressions for the charge, the mass and the entropy given in Eqs. (41), (46) and (47), and using the fact that $f(r_+) = 0$, we obtain a Smarr-type formula as

$$M(S, Q) = -\frac{b^{-\alpha^2}(4S)^{(\alpha^2+1)/2}}{8\pi(\alpha^2-1)} + \frac{(\alpha^2+1)b^{\alpha^2}}{8\pi(\alpha^2-3)}(\Lambda+2\beta^2)(4S)^{(3-\alpha^2)/2} + \frac{b^\gamma\beta Q(\alpha^2+1)^3}{4(\alpha^2-1)^2} \left(\frac{\beta^2 b^{2\gamma}}{16\pi^2 Q^2} \right)^{\frac{1-\gamma}{2\gamma-4}} \left(\frac{1-\alpha^2}{4} \right)^{\frac{2\gamma-3}{2\gamma-4}} \times \left\{ -4(\gamma-2)^2 \left[\Gamma \left(\frac{\alpha^2+5}{4}, \frac{1-\alpha^2}{4} L_W(\zeta) \right) - \Gamma \left(\frac{\alpha^2+5}{4} \right) \right] + (\gamma-1)^2 \left[\Gamma \left(\frac{\alpha^2-3}{4}, \frac{1-\alpha^2}{4} L_W(\zeta) \right) - \Gamma \left(\frac{\alpha^2-3}{4} \right) \right] \right\}, \quad (52)$$

where $\zeta = \frac{\pi^2 Q^2}{S^2 \beta^2}$. If we expand $M(S, Q)$ for large β , we arrive at

$$M(S, Q) = -\frac{b^{-\alpha^2}(4S)^{(\alpha^2+1)/2}}{8\pi(\alpha^2-1)} + \frac{\Lambda(\alpha^2+1)b^{\alpha^2}}{8\pi(\alpha^2-3)}(4S)^{(3-\alpha^2)/2} + 2\pi Q^2 b^{\alpha^2} (4S)^{(-\alpha^2-1)/2} - \frac{8Q^4 \pi^3 (\alpha^2+1)b^{\alpha^2}}{\beta^2(\alpha^2+5)} (4S)^{(-\alpha^2-5)/2} + O\left(\frac{1}{\beta^4}\right), \quad (53)$$

which is exactly the Smarr-type formula obtained for EMd black holes in the limit $\beta^2 \rightarrow \infty$ [32].

Now, if we consider S and Q as a complete set of extensive parameters for the mass $M(S, Q)$, we

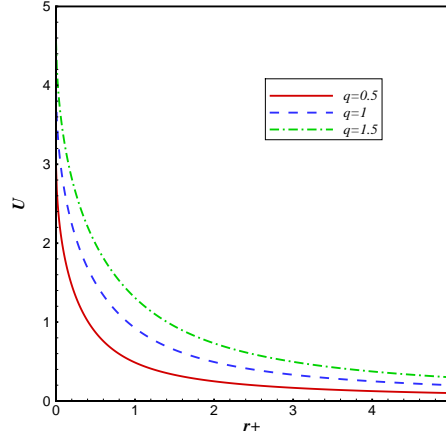


FIG. 14: $U(r_+)$ versus r_+ for $\alpha = 0.5$ and $\beta = 1$.

can define the intensive parameters conjugate to S and Q as

$$T = \left(\frac{\partial M}{\partial S} \right)_Q, \quad U = \left(\frac{\partial M}{\partial Q} \right)_S. \quad (54)$$

After numerical calculations we can show that the intensive quantities calculated by Eq. (54) coincide with Eqs. (42) and (50). Thus, these thermodynamic quantities satisfy the first law of black hole thermodynamics

$$dM = TdS + UdQ. \quad (55)$$

The satisfaction of the first law of thermodynamics for the obtained conserved and thermodynamic quantities, together with the fact that these quantities in two limiting cases, namely in the absence of the dilaton field ($\alpha = 0 = \gamma$), and for large values of the nonlinear parameter ($\beta \rightarrow \infty$), reduce to the known results in the literature [13, 32], indicate that the conserved and thermodynamic quantities obtained in this paper are correct and in agreement with other method such as Euclidean action method [40].

V. CLOSING REMARKS

The pioneering theory of the nonlinear electromagnetic field was proposed by Born and Infeld in 1934 for the purpose of solving various problems of divergence appearing in the Maxwell theory [1]. However, in recent years, other modifications of linear Maxwell's field have been proposed which yield finite self-energy for a charged point particle located at $r = 0$. Among them, are exponential and logarithmic form of Lagrangian for nonlinear electrodynamics. Since the series expansion and

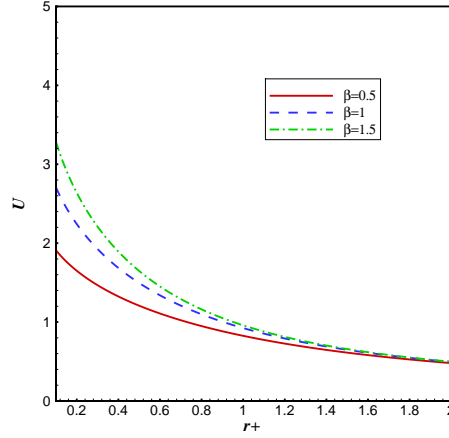


FIG. 15: $U(r_+)$ versus r_+ for $\alpha = 0.5$ and $q = 1$.

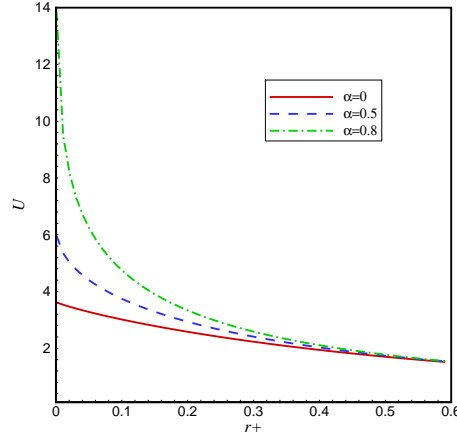


FIG. 16: $U(r_+)$ versus r_+ for $q = 1$ and $\beta = 2$.

the behavior of the solutions of these two Lagrangian, for large value of the nonlinear parameter, are the same as BI Lagrangian, they usually called BI-like Lagrangian in the literatures [13]. Black hole solutions in the presence of BI-like nonlinear electrodynamics have been investigated in [11–13].

In this paper, we have extended the study on the EN electrodynamics by taking into account the dilaton scalar field in the action. We first proposed the suitable Lagrangian for EN electrodynamics coupled to the dilaton field. As far as we know, this is for the first time which Lagrangian (6) is introduced. We have compared in Eqs. (8) and (9) the series expansions of ENd Lagrangian with BId Lagrangian in the limit of large β , and found that they are the same. We have varied the action and obtained the field equations of ENd theory. Then, by making a suitable ansatz (26), we have constructed a new class of charged static and spherically symmetric black hole solutions in

the presence of ENd electrodynamics. In the limiting case where $\beta \rightarrow \infty$, our solutions reduce to EMd black hole solutions [32], while in the absence of the dilaton field, ($\alpha = 0 = \gamma$), they restore charged black holes coupled to EN electrodynamics [13]. We also investigated and compared the behavior of the electric field of three kind of black holes in dilaton gravity, namely EMd, BId and ENd black holes we just found. Although the behavior of the electric fields near the origin depends on the model parameters, however for large r the asymptotic behavior of all of them are exactly the same as linear Maxwell field. Interestingly enough, we found that the electric fields of ENd black hole is finite near the origin and diverges exactly at $r = 0$ depending on the model parameters, however its divergency is much slower than the Maxwell field.

We also investigated the physical properties of the solutions in ample details. The presence of the dilaton field changes the asymptotic behavior of the solutions to be neither flat nor (A)dS. Our solutions can represent black holes with inner and outer horizons, an extreme black hole or naked singularity depending on the model parameters. For fixed value of m , α and q , we found that there is a minimum (extreme) value for nonlinear parameter β_{\min} (β_{ext}), for which we have black hole with a non-extreme horizon provided $\beta \leq \beta_{\min}$, black hole with two horizons for $\beta_{\min} < \beta < \beta_{\text{ext}}$, black hole with an extreme horizon for $\beta = \beta_{\text{ext}}$ and naked singularity for $\beta > \beta_{\text{ext}}$. We computed the mass, entropy, temperature, electric potential of these black holes. We obtained the Smarr-type formula, $M(S, Q)$, and checked that the conserved and thermodynamic quantities obtained for these solutions satisfy the first law of black holes thermodynamics on the horizon.

Finally, we would like to mention that black hole solutions we obtained here are static. Thus, it would be nice to derive rotating black hole/string solutions of these field equations. Besides, it is also interesting to generalize the study to higher dimensions and construct both static and rotating black holes/branes of ENd theory in arbitrary dimensions. In addition, we only considered the exponential nonlinear electrodynamics coupled to the dilaton field, so the case with logarithmic nonlinear electrodynamics in the presence of dilaton field remains to be investigated. Our group are now working on these subjects and the results will be appeared shortly in our future works.

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